# Countable Support Products of Creature Forcings

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## Abstract

Using (mostly) a countable support product of $\limsup$ creature forcings, we construct a ZFC universe in which the cardinal characteristics $\aleph_1$, $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$ and $2^{\aleph_0}$ are all distinct, and moreover distinct from uncountably many other simply definable cardinal characteristics.

## Main Result

Assume CH in the ground model and let $\mathrm{types} := \{\mathrm{nm}, \mathrm{nn}, \mathrm{cn}, \mathrm{ct}\} \cup \bigcup_{\xi<\omega_1}\{\xi\}$ and $\mathrm{types}_{\limsup} := \mathrm{types} \setminus \{\mathrm{nm}\}$. Choose cardinals $\kappa_{\mathrm{nm}} \leq \kappa_{\mathrm{nn}} \leq \kappa_{\mathrm{cn}} \leq \kappa_{\mathrm{ct}}$ as well as a sequence of cardinals $\langle \kappa_\xi \rangle_{\xi<\omega_1}$ with $\kappa_{\mathrm{nm}} \leq \kappa_\xi \leq \kappa_{\mathrm{nn}}$ such that for each $t \in \mathrm{types}$, $\kappa_t^{\aleph_0} = \kappa_t$; also choose a series of functions $\langle f_\xi, g_\xi \rangle_{\xi<\omega_1}$ with $g_\xi < f_\xi$ and with sufficiently different asymptotic growth. There are natural $\limsup$ creature forcings $\mathbb{Q}_t$ for each $t \in \mathrm{types}_{\limsup}$ and a $\liminf$ creature forcing $\mathbb{Q}_{\mathrm{nm},\kappa_{\mathrm{nm}}}$ such that

$$\mathbb{Q} := \prod_{t \in \mathrm{types}_{\limsup}} \mathbb{Q}_t^{\kappa_t} \times \mathbb{Q}_{\mathrm{nm},\kappa_{\mathrm{nm}}}$$

(where all products and powers have countable support) forces:

- $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$,
- $\mathrm{non}(\mathcal{M}) = \kappa_{\mathrm{nm}}$,
- $\mathrm{non}(\mathcal{N}) = \kappa_{\mathrm{nn}}$,
- $\mathrm{cof}(\mathcal{N}) = \kappa_{\mathrm{cn}}$,
- $2^{\aleph_0} = \kappa_{\mathrm{ct}}$, and
- $\mathfrak{c}_{f_\xi,g_\xi} = \kappa_\xi$ for all $\xi < \omega_1$.

Moreover, $\mathbb{Q}$ preserves all cardinals and cofinalities.

## Methods and Properties

The forcing $\mathbb{Q}_{\mathrm{nm},\kappa_{\mathrm{nm}}}$ as well as each of the other forcing notions $\mathbb{Q}_t$ is a creature forcing, which means the conditions are infinite sequences $(c_0, c_1, \ldots)$ of finite objects, so-called "creatures", and each creature carries some information about a finite segment of the generic real. Hence $\mathbb{Q}$ has a natural level structure. We remark on a few important facets of the proofs:

- The set of conditions with only finitely many nontrivial creatures at level $\ell$ is dense. Moreover, so is the set of modest conditions with at most one nontrivial creature at each level
- Each creature is much "bigger" than all creatures on lower levels, providing a kind of Ramsey property and "thinning out" procedures.
- $p \in \mathbb{Q}$ essentially decides an ordinal name $\dot{\tau}$ if there is a level $m$ such that for each $m$-initial segment $\eta$ of $p$, $\eta^\frown p^{>m}$ already decides $\dot{\tau}$.
- Given $p \in \mathbb{Q}$ and an ordinal name $\dot{\tau}$, we can find $q \leq p$ essentially deciding $\dot{\tau}$. Moreover, given $h \in \omega$, we can do this in such a way that $q^{\leq h} = p^{\leq h}$ and such that the creatures above $h$ remain sufficiently large; we call this pure decision. This is straightforward for the $\limsup$ forcings, but the proof for the $\liminf$ forcing is more sophisticated and involves the so-called halving construction.
- Iterating pure decision allows for a fusion construction essentially deciding countably many ordinal names, hence $\mathbb{Q}$ satisfies Axiom A.
- In a similar way, we prove that $\mathbb{Q}$ has continuous reading (i. e. every real is a continuous image of (countably many) generic reals) as well as stronger versions of reading, in the spirit of uniform or Lipschitz continuity.

## Results in Cichoń's Diagram

For any given ideal $\mathcal{I}$ of some base set $X$, the following four cardinal characteristics are of interest:

$$\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\} \qquad \mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = X\}$$

$$\mathrm{non}(\mathcal{I}) := \min\{|Y| \mid Y \subseteq X, Y \notin \mathcal{I}\} \qquad \mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\colon B \subseteq A\}$$

Cichoń's diagram collects the ZFC-provable inequalities between the cardinal characteristics of the following three ideals:

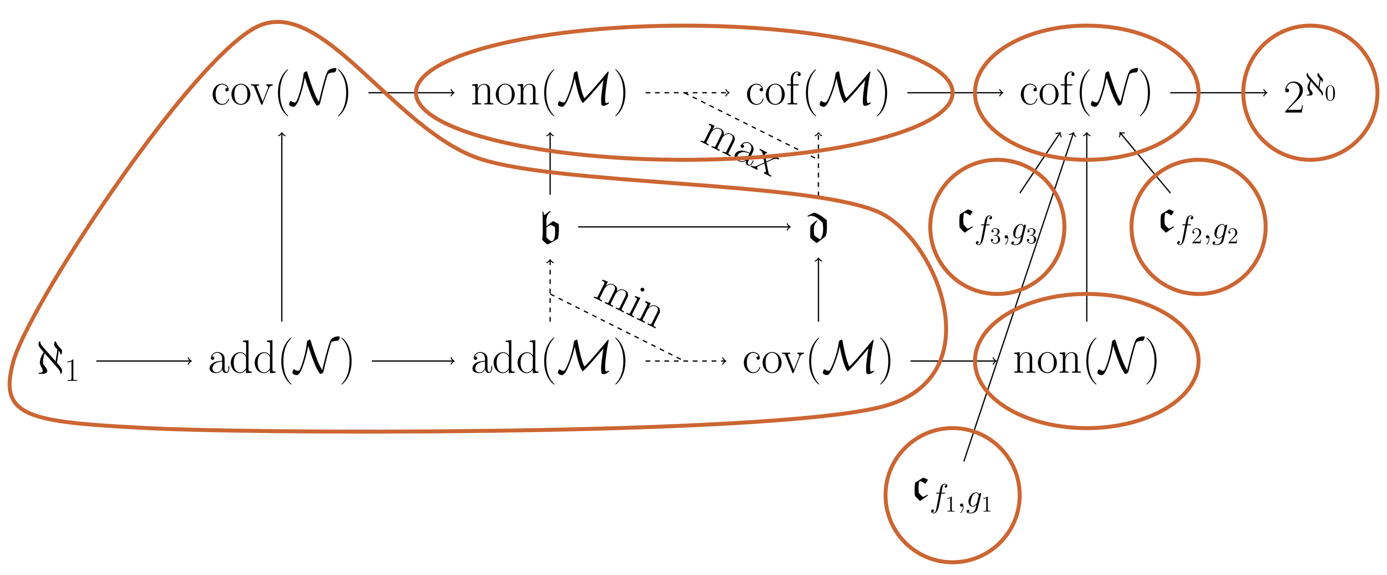
- the ideal $\mathcal{N} := \{A \subseteq 2^\omega \mid \lambda(A) = 0\}$ of Lebesgue null sets
- the ideal $\mathcal{M} := \{A \subseteq \omega^\omega \mid A = \bigcup_{n\in\omega} A_n, A_n \text{ nowhere dense}\}$ of meagre sets
- the ideal $\mathcal{K} := \{A \subseteq \omega^\omega \mid \exists f \in \omega^\omega\, \forall x \in A\colon x \leq f\}$ generated by the compact sets

For $\mathcal{K}$, it is known that $\mathfrak{b} := \mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K})$ and $\mathfrak{d} := \mathrm{cov}(\mathcal{K}) = \mathrm{cof}(\mathcal{K})$, though $\mathfrak{b}$ and $\mathfrak{d}$ are more commonly defined as follows:

$$\mathfrak{b} := \min\{|A| \mid A \subseteq \omega^\omega, \forall g \in \omega^\omega\, \exists f \in A\colon f \not\leq^* g\}$$

$$\mathfrak{d} := \min\{|A| \mid A \subseteq \omega^\omega, \forall g \in \omega^\omega\, \exists f \in A\colon g \leq^* f\}$$

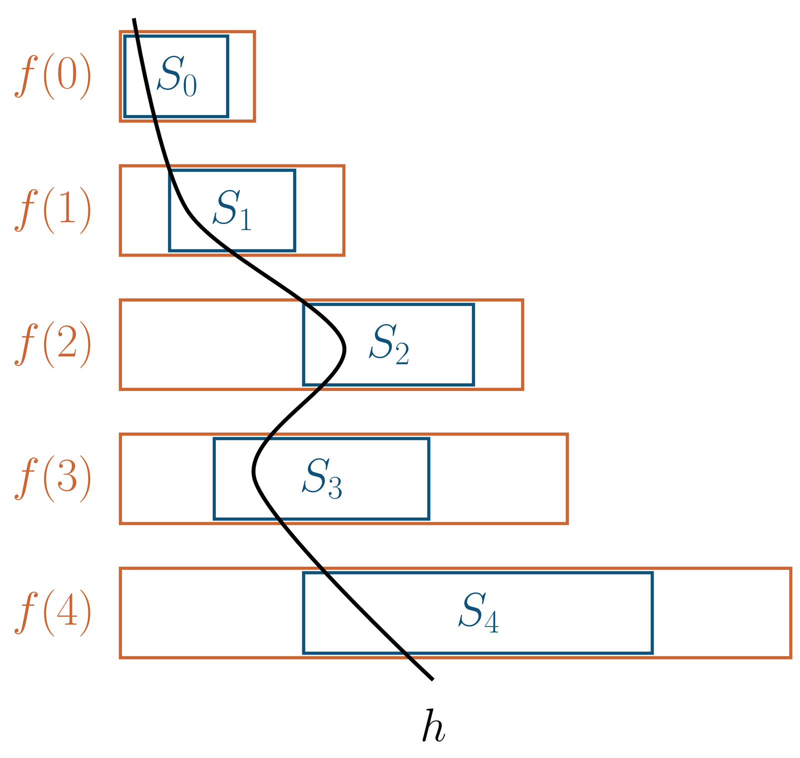
Shown here are the results of our forcing $\mathbb{Q}$ for these cardinal characteristics; cardinals which are forced to be equal are grouped together, and each such group can be forced to be different from the others.




Cichoń's diagram as well as a few exemplary $\mathfrak{c}_{f,g}$

## $(f, g)$-Slaloms and $(f, g)$-Covering Families

Given $f, g \in \omega^\omega$ diverging to infinity such that $0 < g < f$, we call $S := \langle S_k \rangle_{k\in\omega} \in ([\omega]^{<\omega})^\omega$ an $(f, g)$-slalom if $S_k \subseteq f(k)$ and $|S_k| \leq g(k)$ for all $k \in \omega$. We say a family of $(f, g)$-slaloms $\mathcal{S}$ is $(f, g)$-covering if for all $h \in \prod_{k\in\omega} f(k)$ there is an $S \in \mathcal{S}$ such that $h \in S$ (i. e. $h(k) \in S_k$ for all $k \in \omega$).




Initial segment of an $(f, g)$-slalom $S$ and a function $h \in S$

We then define the cardinal characteristic $\mathfrak{c}_{f,g}$ as

$$\mathfrak{c}_{f,g} := \min\{|\mathcal{S}| \mid \mathcal{S} \text{ is an } (f, g)\text{-covering family}\}.$$

If we extend the definition to allow $f \in (\omega + 1)^\omega$ and set $w$ to be the constant $\omega$-function, it is ZFC-provable that $\mathfrak{c}_{f,g} \leq \mathfrak{c}_{w,g} = \mathrm{cof}(\mathcal{N})$ for any such $f, g$.

## Bibliography and History

This work is mostly based on [FGKS] and [GS]. Blass in [B] posed the question of whether there might be a reasonable classification of all uniform $\mathbf{\Pi}^0_1$ characteristics, which Goldstern and Shelah answered in the negative in [GS], providing a universe in which (at least) $\aleph_1$ many cardinals appear as $\mathbf{\Pi}^0_1$ characteristics. Fischer, Goldstern, Kellner and Shelah in [FGKS] constructed a forcing to separate most of the right-hand side of Cichoń's diagram, using a technique almost, but not quite, entirely unlike a countable support product.


- [B] Andreas Blass. Simple Cardinal Characteristics of the Continuum. In: Haim Judah (Ed.). *Set Theory of the Reals*. Israel Math. Conf. Proc., vol. 6, pp. 63–90 (1993).
- [FGKS] Arthur Fischer, Martin Goldstern, Jakob Kellner and Saharon Shelah. Creature Forcing and Five Cardinal Characteristics in Cichoń's Diagram. *Arch. Math. Logic* (2017).
- [GS] Martin Goldstern and Saharon Shelah. Many Simple Cardinal Invariants. *Arch. Math. Logic*, 32(3):203–221 (1993).


For further background on cardinal characteristics, see the survey articles by Tomek Bartoszyński and by Andreas Blass in the *Handbook of Set Theory*.

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