Halfway New Cardinal Characteristics

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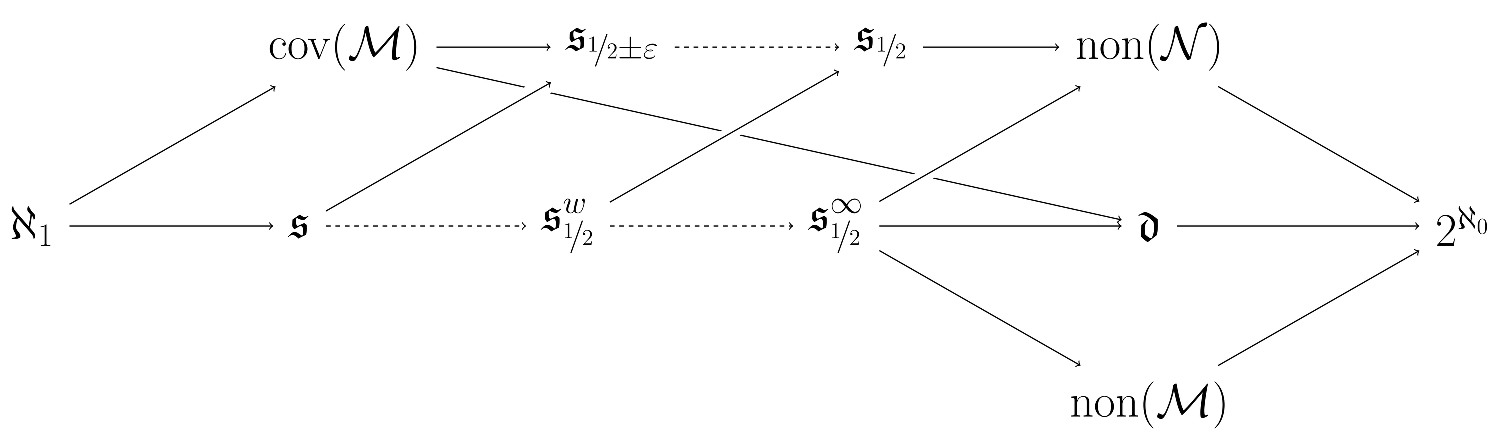
Abstract

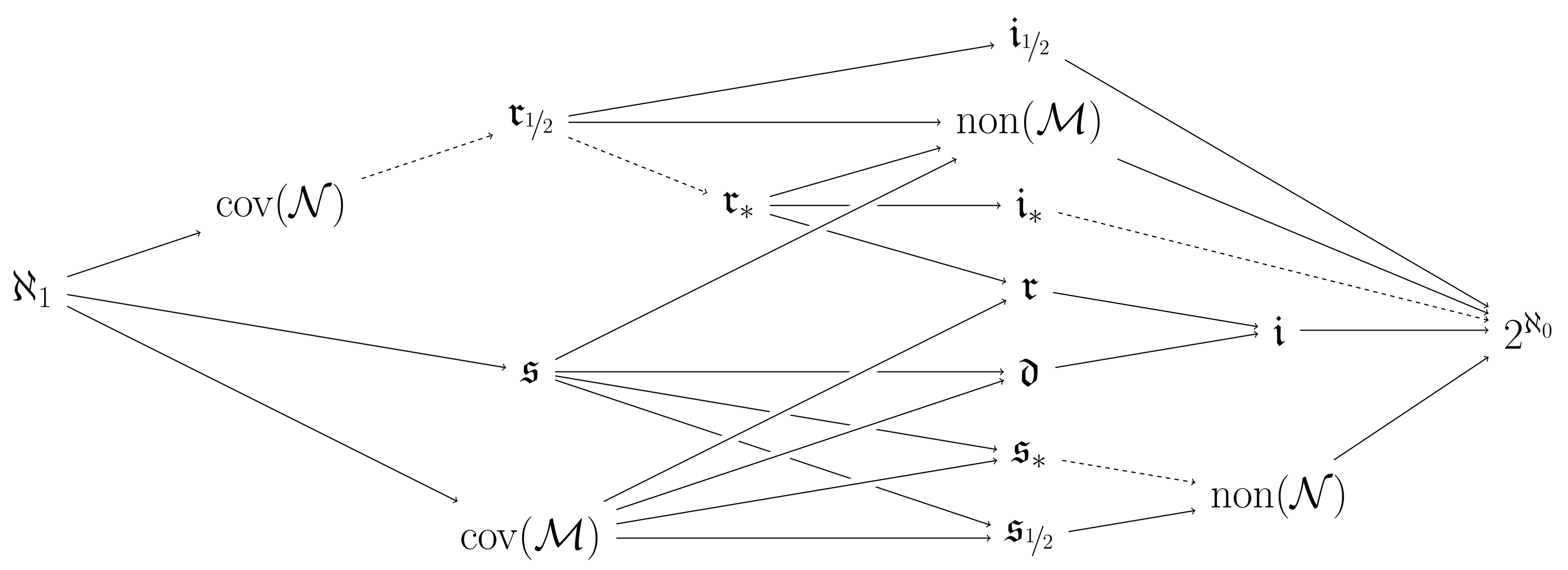
We introduce several cardinal characteristics related to the splitting number $\mathfrak{s}$, the reaping number $\mathfrak{r}$ and the independence number $\mathfrak{i}$ and prove bounds and consistency results.

Main Results

Our results are summarised in the following two diagrams, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".



The ZFC-provable and/or consistent inequalities between $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$ and other well-known cardinal characteristics.



The ZFC-provable and/or consistent inequalities between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$, $\mathfrak{r}_{1/2}$, $\mathfrak{r}_*$, $\mathfrak{s}_{1/2}$, $\mathfrak{s}_*$ and other well-known cardinal characteristics.

Proof Methods

Most of the results we show in our paper rely on properties of Cohen forcing, basic combinatorial arguments, relatively simple probabilistic arguments or previously known consistency results, with three exceptions:

- The proof of $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$ uses the standard countable support product creature forcing construction to increase $\mathrm{non}(\mathcal{N})$ together with some moderately sophisticated probabilistic arguments to ensure the parameters of the creature forcing are chosen sufficiently large.
- The proofs of $\mathrm{Con}(\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2})$ and $\mathrm{Con}(\mathfrak{r}_* < \mathfrak{i}_*)$ use the template iteration from Brendle's *Mad Families and Iteration Theory* with only minor modifications to the isomorphism-of-names step of the argument.
- The proof of $\mathrm{Con}(\mathfrak{i}_{1/2} < \mathfrak{i})$ is analogous to the classical proof of $\mathrm{Con}(\aleph_1 = \mathfrak{a} < 2^{\aleph_0})$ using λ many Cohen reals, although it also involves a fair bit of combinatorics.

Preliminaries

Let $X \in [\omega]^\omega$ and $0 < n < \omega$. Recall the following concepts from number theory:

- $d_n(X) := \frac{|X \cap n|}{n}$, the initial density of X up to n,
- $\underline{d}(X) := \liminf_{n\to\infty}(d_n(X))$, the lower density of X,
- $\bar{d}(X) := \limsup_{n\to\infty}(d_n(X))$, the upper density of X, and
- $d(X) := \lim_{n\to\infty}(d_n(X))$, the (asymptotic) density of X (in case of convergence).

We call a set $X \in [\omega]^\omega$ moderate if $\underline{d}(X) > 0$ as well as $\bar{d}(X) < 1$.

Definitions

Let $S, X \in [\omega]^\omega$. We define the following relations:

$$S \mid_{1/2} X \Leftrightarrow \text{“}S \text{ bisects } X \text{ (in the limit)”}$$
$$\Leftrightarrow \lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}$$
$$S \mid_{1/2\pm\varepsilon} X \Leftrightarrow \text{“}S\ \varepsilon\text{-almost bisects } X\text{”}$$
$$\Leftrightarrow \forall^\infty n < \omega\colon \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$
$$S \mid^w_{1/2} X \Leftrightarrow \text{“}S \text{ weakly bisects } X\text{”}$$
$$\Leftrightarrow \forall \varepsilon > 0\ \exists^\infty n < \omega\colon \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$
$$S \mid^\infty_{1/2} X \Leftrightarrow \text{“}S \text{ bisects } X \text{ infinitely often”}$$
$$\Leftrightarrow \exists^\infty n < \omega\colon \frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}$$

We denote the least cardinality of an $\mathcal{S} \subseteq [\omega]^\omega$ with the property that for each $X \in [\omega]^\omega$, there is an $S \in \mathcal{S}$ such that S bisects X, ε-almost bisects X, weakly bisects X or bisects X infinitely often by $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$, respectively.

$\mathcal{I}_* \subseteq [\omega]^\omega$ is statistically independent ($*$-independent) if all $X \in \mathcal{I}_*$ are moderate and for any finite $\mathcal{E} \subseteq \mathcal{I}_*$, we have

$$\lim_{n\to\infty} \left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)} \right) = 1.$$

$\mathcal{I}_{1/2} \subseteq [\omega]^\omega$ is $1/2$-independent if for any finite $\mathcal{A} \mathbin{\dot\cup} \mathcal{B} \subseteq \mathcal{I}_{1/2}$, we have

$$d\left(\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)\right) = 1/2^{|\mathcal{A}|+|\mathcal{B}|}.$$

$\mathcal{R}_* \subseteq [\omega]^\omega$ is statistically reaping ($*$-reaping) if there is no moderate $S \in [\omega]^\omega$ such that

$$\forall X \in \mathcal{R}_*\colon \lim_{n\to\infty} \left(\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} \right) = 1.$$

$\mathcal{R}_{1/2} \subseteq [\omega]^\omega$ is $1/2$-reaping if there is no $S \in [\omega]^\omega$ bisecting all $R \in \mathcal{R}_{1/2}$.

$\mathcal{S}_* \subseteq [\omega]^\omega$ is statistically splitting ($*$-splitting) if for each $X \in [\omega]^\omega$, there is a moderate $S \in \mathcal{S}_*$ such that

$$\lim_{n\to\infty} \left(\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} \right) = 1.$$

We denote the least cardinality of a maximal $*$-independent, maximal $1/2$-independent, $*$-reaping, $1/2$-reaping, $*$-splitting family by $\mathfrak{i}_*$, $\mathfrak{i}_{1/2}$, $\mathfrak{r}_*$, $\mathfrak{r}_{1/2}$, $\mathfrak{s}_*$, respectively.

Open Questions

- Which of the following statements are true?

$\mathrm{Con}(\mathfrak{d} < \mathfrak{s}_{1/2(\pm\varepsilon)})$	or	$\mathfrak{s}_{1/2(\pm\varepsilon)} \leq \mathfrak{d}$
$\mathrm{Con}(\mathfrak{s} < \mathfrak{s}^w_{1/2})$	or	$\mathfrak{s} = \mathfrak{s}^w_{1/2}$
$\mathrm{Con}(\mathfrak{s}^w_{1/2} < \mathfrak{s}^\infty_{1/2})$	or	$\mathfrak{s}^w_{1/2} = \mathfrak{s}^\infty_{1/2}$
$\mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}_{1/2})$	or	$\mathfrak{s}_{1/2\pm\varepsilon} = \mathfrak{s}_{1/2}$

- Can characteristics in the upper row of the diagram consistently be smaller than ones in the lower row? Specifically, which of the following statements are true?

$\mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^w_{1/2})$	or	$\mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^w_{1/2}$
$\mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^\infty_{1/2})$	or	$\mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^\infty_{1/2}$
$\mathrm{Con}(\mathfrak{s}_{1/2} < \mathfrak{s}^\infty_{1/2})$	or	$\mathfrak{s}_{1/2} \geq \mathfrak{s}^\infty_{1/2}$

- Does $\mathfrak{s}_{1/2\pm\varepsilon}$ depend on ε?
- Is it consistent that $\mathfrak{i}_* < 2^{\aleph_0}$?
- Which relations between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$ and $\mathfrak{i}$ are true or consistent?
- Are there any smaller upper bounds for $\mathfrak{i}_{1/2}$ and $\mathfrak{i}_*$?
- Which relations between $\mathfrak{s}_{1/2}$ and $\mathfrak{s}_*$ are true or consistent?
- Which of the following statements are true?

$\mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2})$	or	$\mathrm{cov}(\mathcal{N}) = \mathfrak{r}_{1/2}$
$\mathrm{Con}(\mathfrak{r}_{1/2} < \mathfrak{r}_*)$	or	$\mathfrak{r}_{1/2} = \mathfrak{r}_*$
$\mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N}))$	or	$\mathfrak{s}_* = \mathrm{non}(\mathcal{N})$